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VALUATIONS ON MANIFOLDS AND RUMIN COHOMOLOGY

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Abstract

Smooth valuations on manifolds are studied by establishing a link with the Rumin-de Rham complex of the co-sphere bundle. Several operations on differential forms induce operations on smooth valuations: signature operator, Rumin-Laplace operator, Euler-Verdier involution and derivation operator. As an application, Alesker's Hard Lefschetz Theorem for even translation invariant valuations on a finite-dimensional Euclidean space is generalized to all translation invariant valuations. The proof uses Kähler identities, the Rumin-de Rham complex and spectral geometry.

Introduction

Let V be an n-dimensional vector space and denote by $\mathcal{K}(V)$ the space of compact convex sets in V. A *convex valuation* on V is a map $\Psi : \mathcal{K}(V) \to \mathbb{R}$ with the following property (Euler additivity): if $K_1, K_2, K_1 \cup K_2 \in \mathcal{K}(V)$, then

$$\Psi(K_1 \cap K_2) + \Psi(K_1 \cup K_2) = \Psi(K_1) + \Psi(K_2).$$

By Hadwiger's theorem, the space of motion invariant and continuous (with respect to Hausdorff topology) valuations is a valuation space of dimension $n+1$. In contrast to this, the space $\mathrm{Val}(V)$ of translation invariant continuous valuations is an infinite-dimensional Fréchet-space. The algebraic structures underlying this space were studied by Alesker using deep representation-theoretic tools. There is a natural $GL(V)$-action on $\mathrm{Val}(V)$. The subspace of $GL(V)$-smooth vectors is denoted by $\mathrm{Val}^{sm}(V)$. From his Irreducibility Theorem, Alesker deduces in [**6**] that a smooth valuation can be represented by integration of a differential form on the co-sphere bundle against conormal cycles of compact convex sets. He also defined and studied smooth valuations on an arbitrary manifold [**6**], [**7**], replacing convex sets (which would not make sense on an arbitrary manifold) by *differentiable polyhedra*.

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In this paper, we work in a slighly different setting, namely that of an *analytic-geometric category* $\mathcal{C}$ in the sense of [**18**]. The definition of such a category is recalled in Section 1. We denote by $\mathcal{C}_b(M)$ the subset of relatively compact sets in $\mathcal{C}(M)$. The conormal cycle of a set $X \in \mathcal{C}_b(M)$ is denoted by $\operatorname{cnc}(X)$. If M is endowed with a Riemannian metric, then $\operatorname{nc}(X)$ denotes the normal cycle of X (compare [**10**]).

Definition 0.1.

- A $\mathcal{C}$-valuation on a real-analytic manifold M is a map $\Psi : \mathcal{C}_b(M) \to \mathbb{R}$ such that
$$\Psi(X \cup Y) + \Psi(X \cap Y) = \Psi(X) + \Psi(Y); \quad X, Y \in \mathcal{C}_b(M).$$
- A $\mathcal{C}$-valuation Ψ is called *smooth* if there exist a smooth differential $n-1$-form ω on S^*M and a smooth n-form ϕ on M such that
$$\Psi(X) = \Psi_{(\omega,\phi)}(X) := \operatorname{cnc}(X)(\omega) + \int_X \phi$$
for all $X \in \mathcal{C}_b(M)$.

We refer to [**7**] for equivalent conditions and for further information on smooth valuations. The space of smooth valuations on M is denoted by $\mathcal{V}^\infty(M)$.

A given smooth valuation can be represented by different pairs (ω, ϕ). Our first main theorem describes the kernel of the map $(\omega, \phi) \mapsto \Psi_{(\omega,\phi)}$ in terms of the Rumin operator $D : \Omega^{n-1}(S^*M) \to \Omega^n(S^*M)$. The definition of this second-order differential operator is recalled in Section 1.

Theorem 1. *Let M be an oriented real-analytic manifold of dimension n and $\mathbf{\Psi} : \Omega^{n-1}(S^*M) \oplus \Omega^n(M) \to \mathcal{V}^\infty(M), (\omega, \phi) \mapsto \Psi_{(\omega,\phi)}$ the map introduced above. Then $(\omega, \phi) \in \ker(\mathbf{\Psi})$ if and only if*

1) *$D\omega + \pi^*\phi = 0$ and*
2) *$\int_{S_p^*M} \omega = 0$ for all $p \in M$.*

Here $\pi : S^*M \to M$ is the canonical projection and $S_p^*M = \pi^{-1}(p)$.

In the important case where M is a real vector space, the ϕ-part is not needed (Corollary 1.6). The condition $D\omega = 0$ is then equivalent to saying that, up to some vertical form, ω is d-closed, while the second condition means that the de Rham cohomology class of ω is trivial. The kernel of the map $\omega \mapsto \Psi_{(\omega,0)}$ is thus generated by vertical and exact forms.

We have stated Theorem 1 for smooth $\mathcal{C}$-valuations. However, as the proof will show, it also holds for other kinds of smooth valuations. This is the case for convex valuations (on a real vector space V) and valuations on differentiable polyhedra on an arbitrary manifold [**7**]. What is needed is that the considered class of sets is large enough to admit local variations. Therefore the proof does *not* work for valuations on polytopes (but one can use a density argument to show that the analogous statement still holds).

Using Theorem 1, we will introduce several operations on smooth valuations. First of all, the natural involution on S^*M induces an involution, called *Euler-Verdier involution*, on the space of smooth valuations on M (it was previously studied by Alesker [**7**]). If M is endowed with a Riemannian metric, there are three more operators acting on smooth valuations. They are induced by the signature operator, the Laplace operator and the Lie derivation with respect to the Reeb vector field on S^*M. This last operator, called *derivation operator* and denoted by $\mathfrak{L}$, was studied in the Euclidean case by Alesker [**3**]. In this case, $\mathfrak{L}\Psi(K) = \frac{d}{dt}\big|_{t=0} \Psi(K+tB)$, where $K+tB$ denotes the tube of radius t around a compact convex set K (compare Proposition 3.4).

Before we state our second main theorem, we need to recall a result of McMullen. A translation invariant valuation on an n-dimensional vector space V is called *homogeneous of degree k* if $\Psi(tK) = t^k\Psi(K)$ for all compact convex sets K and all $t > 0$. With $\mathrm{Val}_k(V)$ being the subspace of valuations of degree k, McMullen's result is the decomposition

$$\mathrm{Val}(V) = \bigoplus_{k=0}^{n} \mathrm{Val}_k(V).$$

It is easily checked that $\mathfrak{L}$ decreases the degree of a valuation by 1.

Theorem 2 (Hard Lefschetz Theorem)**.** *Let $\frac{n}{2} < k \leq n$. Then*

$$\mathfrak{L}^{2k-n} : \mathrm{Val}_k^{sm}(V) \to \mathrm{Val}_{n-k}^{sm}(V)$$

is an isomorphism. In particular, $\mathfrak{L} : \mathrm{Val}_k^{sm}(V) \to \mathrm{Val}_{k-1}^{sm}(V)$ is injective for $k \geq \frac{n+1}{2}$ and surjective for $k \leq \frac{n+1}{2}$.

In the special case of *even* translation invariant valuations (a valuation is even if $\Psi(-K) = \Psi(K)$ for all K), the theorem was proved by Alesker using representation theory of $GL(V)$. He also gave the name *Hard Lefschetz theorem*, stressing the formal analogy with the Hard Lefschetz Theorem for compact Kähler manifolds. Our proof shows that there is more than just an analogy, since it relies on the geometry of the Kähler manifold $V \times (V \setminus \{0\})$ and the Kähler identities on it.

Let G be any subgroup of $O(V)$. We denote by $\mathrm{Val}^G(V)$ the space of G-invariant, translation invariant valuations and by $\mathrm{Val}^{sm}(V)^G$ the space of smooth G-invariant, translation invariant valuations.

Corollary 0.2.

1) *Let G be any subgroup of $O(V)$. Then*

$$\mathfrak{L}^{2k-n} : \mathrm{Val}_k^{sm}(V)^G \to \mathrm{Val}_{n-k}^{sm}(V)^G$$

is an isomorphism for $\frac{n}{2} < k \leq n$.

2) *Suppose G is compact and acts transitively on $S(V)$. Then* $\mathrm{Val}_k^G(V)$ *is finite-dimensional and* $h_k^G := \dim \mathrm{Val}_k^G(V)$ *satisfy the Lefschetz inequalities*

$$h_k^G \leq h_{k+1}^G \text{ for } k < \frac{n}{2}$$

and

$$h_k^G = h_{n-k}^G \text{ for } 0 \leq k \leq n.$$

The finite-dimensionality of $\mathrm{Val}_k^G(V)$ and the equality in the second part were obtained by Alesker [**1**], [**5**]. The inequality was conjectured by Alesker [**5**] and proved under the additional assumption $-Id \in G$.

The paper is organized as follows. In Section 1, we introduce smooth valuations on manifolds. We recall the definition of an Analytic-Geometric Category and the normal cycle construction. Then we prove Theorem 1 and state some corollaries. In Section 2, we study several natural operations on smooth valuations: Euler-Verdier involution, signature and Laplace operator, and derivation operator. The relation between translation invariant smooth valuations on a finite-dimensional Euclidean space V and translation invariant differential forms on the sphere bundle SV is the subject of Section 3. We also recall some results of McMullen and Alesker concerning translation invariant valuations. The proof of Theorem 2 is contained in Section 4.

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1. Valuations on manifolds

The following definition is taken from [**18**].

Definition 1.1. An analytic-geometric category $\mathcal{C}$ is a set of pairs (M, X) such that

1) M is a real-analytic manifold and X a subset of M;
2) for fixed M, the set $\mathcal{C}(M) := \{X \subseteq M : (M, X) \in \mathcal{C}\}$ is a Boolean algebra which contains M,
3) if $(M, X) \in \mathcal{C}$ then $(M \times \mathbb{R}, X \times \mathbb{R}) \in \mathcal{C}$;
4) for each proper analytic map $f : M \to N$ and $X \in \mathcal{C}(M)$, $f(X) \in \mathcal{C}(N)$;
5) if $X \subseteq M$, (U_i) an open covering of M, then $X \in \mathcal{C}(M)$ if and only if $X \cap U_i \in \mathcal{C}(U_i)$ for each i;
6) the bounded sets in $\mathcal{C}(\mathbb{R})$ are exactly finite unions of bounded intervals.

The basic example of an analytic-geometric category is that of subanalytic sets, but there are many others. The reader who is not familiar with analytic-geometric categories may think of subanalytic sets throughout this paper.

Fix an analytic-geometric category $\mathcal{C}$ and an oriented, real-analytic manifold M. Let $\mathcal{C}_b(M)$ be the set of relatively compact subsets $X \in \mathcal{C}(M)$.

Definition 1.2. A $\mathcal{C}$-valuation on M is a map

$$\Psi : \mathcal{C}_b(M) \to \mathbb{R}$$

with the following property (Euler additivity): if $X, Y \in \mathcal{C}_b(M)$, then

$$\Psi(X \cap Y) + \Psi(X \cup Y) = \Psi(X) + \Psi(Y).$$

If it is clear from the context that we are considering convex or $\mathcal{C}$-valuations, we will just write *valuation*.

Without further assumptions, not much can be said about valuations. In the case of convex valuations, one classically has two types of assumptions. The first one is continuity in the Hausdorff topology. The second one is invariance under the group of Euclidean motions or some sufficiently large subgroup (e.g., the group of translations or rotations).

In the case of $\mathcal{C}$-valuations on an arbitrary oriented real analytic n-dimensional manifold M, we will make throughout the paper the hypothesis that Ψ is smooth in the sense defined below.

Let us introduce some notation first. The co-sphere bundle S^*M of M is the quotient $(T^*M \setminus \{0\})/\mathbb{R}_+$. It will be convenient to consider S^*M as the set of pairs (p, P) with $p \in M$ and $P \subset T_pM$ an oriented hyperplane. The conormal cycle $\operatorname{cnc}(X)$ of $X \in \mathcal{C}_b(M)$ was constructed in [**10**]. It is an integral Legendrian $n-1$-cycle on S^*M.

Definition 1.3. A valuation Ψ is called *smooth* if there exist a smooth differential $n-1$-form ω on S^*M and a smooth n-form ϕ on M such that

$$\Psi(X) = \operatorname{cnc}(X)(\omega) + \int_X \phi$$

for all $X \in \mathcal{C}_b(M)$.

The vector space of smooth valuations on M is denoted by $\mathcal{V}^\infty(M)$.

Example.

- If M is endowed with a real-analytic metric, then one can define a sequence of Lipschitz-Killing invariants $\Psi_0(X), \ldots, \Psi_n(X)$ of $X \in \mathcal{C}_b(M)$. They are smooth valuations (compare [**9**]).
- Let $M = V$ be an Euclidean vector space with $W \subset V$ a linear k-subspace. Define a valuation Ψ on $\mathcal{C}_b(V)$ by setting $\Psi(X) = \operatorname{vol}_k((\pi_W)_* 1_X)$, where $\pi_W : V \to W$ denotes orthogonal projection and $(\pi_W)_* 1_X$ is the push-forward of the characteristic function of

X to W, i.e., the constructible function on W given by $w \mapsto \chi(\pi_W^{-1}(w) \cap X)$. Then Ψ is a non-smooth valuation. Indeed, let U be a k-dimensional linear subspace and X the unit ball in U. Then $\Psi(X) = |\cos(W,U)|$ (compare e.g., [**4**]) but the map $\mathrm{Gr}_k(V) \to \mathbb{R}, U \mapsto |\cos(W,U)|$ is clearly not smooth.
- Let V be an Euclidean vector space and G a compact subgroup of $O(V)$ acting transitively on the unit sphere $S(V)$. Then each continuous G-invariant, translation invariant valuation is smooth ([**3**], Corollary 1.1.3 and [**6**]).

Let ω be a smooth $n-1$-form on S^*M and ϕ a smooth n-form on M. The valuation $X \mapsto \mathrm{cnc}(X)(\omega) + \int_X \phi$ will be denoted by $\Psi_{(\omega,\phi)}$. By definition, we get a surjective linear map

$$\mathbf{\Psi} : \Omega^{n-1}(S^*M) \oplus \Omega^n(M) \to \mathcal{V}^\infty(M), (\omega,\phi) \mapsto \Psi_{(\omega,\phi)}.$$

The kernel of $\mathbf{\Psi}$ is not trivial. For instance, if ω is an exact form, then $\mathbf{\Psi}(\omega, 0) = 0$, since conormal cycles are closed. Similarly, if ω vanishes on the contact distribution, then $\mathbf{\Psi}(\omega,0) = 0$, since conormal cycles are Legendrian. Our first main theorem characterizes the kernel of $\mathbf{\Psi}$. Before proving it, we have to recall some facts about Rumin cohomology [**17**].

Let (N,Q) be a contact manifold of dimension $2n-1$. For simplicity, we suppose that there exists a global contact form α, i.e., $Q = \ker\alpha$. This global contact form is not unique, since multiplication by any non-zero smooth function on N yields again a contact form. However, the following spaces only depend on (N,Q) and not on the particular choice of α:

$$\begin{aligned}
&\Omega^k := \Omega^k(N);\\
&\mathcal{I}^k = \{\omega \in \Omega^k : \omega = \alpha \wedge \xi + d\alpha \wedge \psi, \xi \in \Omega^{k-1}, \psi \in \Omega^{k-2}\};\\
&\mathcal{J}^k = \{\omega \in \Omega^k : \alpha \wedge \omega = d\alpha \wedge \omega = 0\}.
\end{aligned}$$

Since $d\mathcal{I}^k \subset \mathcal{I}^{k+1}$, there exists an induced operator $d_Q : \Omega^k/\mathcal{I}^k \to \Omega^{k+1}/\mathcal{I}^{k+1}$.

Similarly, $d\mathcal{J}^k \subset \mathcal{J}^{k+1}$ and the restriction of d to $\mathcal{J}^k$ yields an operator $d_Q : \mathcal{J}^k \to \mathcal{J}^{k+1}$.

In the middle dimension, there is a further operator, which we will call the *Rumin operator*, which is defined as follows. Let $\omega \in \Omega^{n-1}$. There exists $\xi \in \Omega^{n-2}$ such that $d(\omega + \alpha \wedge \xi) \in \mathcal{J}^n$, and this last form, which is unique, is denoted by $D\omega$. It can be checked that $D|_{\mathcal{I}^{n-1}} = 0$, hence there is an induced operator $D : \Omega^{n-1}/\mathcal{I}^{n-1} \to \mathcal{J}_n$.

The Rumin complex of the contact manifold (N, Q) is given by

$$0 \to C^\infty(N) \stackrel{d_Q}{\to} \Omega^1/\mathcal{I}^1 \stackrel{d_Q}{\to} \ldots \stackrel{d_Q}{\to} \Omega^{n-2}/\mathcal{I}^{n-2} \stackrel{d_Q}{\to} \Omega^{n-1}/\mathcal{I}^{n-1} \stackrel{D}{\to} \mathcal{J}_n \stackrel{d_Q}{\to}$$
$$\stackrel{d_Q}{\to} \mathcal{J}_{n+1} \stackrel{d_Q}{\to} \ldots \stackrel{d_Q}{\to} \mathcal{J}_{2n-1} \to 0.$$

The cohomology of this complex is called *Rumin cohomology* and denoted by $H_Q^*(N, \mathbb{R})$. By [**17**], there exists a natural isomorphism between Rumin cohomology and de Rham cohomology:

$$H_Q^*(N, \mathbb{R}) \xrightarrow{\cong} H_{dR}^*(N, \mathbb{R}). \tag{1}$$

In the middle dimension, this isomorphism can be described as follows. Let $[\omega] \in H_Q^{n-1}(N, \mathbb{R})$. Then $D\omega = 0$, which means that there exists a unique form $\omega' = \omega + \alpha \wedge \xi$ with $d\omega' = 0$. Then ω' defines an element $[\omega'] \in H_{dR}^{n-1}(N, \mathbb{R})$.

Let us return to our special situation where $N = S^*M$. The contact plane at a point (p, P) is given by $(d\pi_p)^{-1}(P)$ (here $\pi : S^*M \to M$ is the natural projection). We fix a global contact form α, i.e., a 1-form whose kernel is the contact distribution.

Proof of Theorem 1.

1) Suppose $D\omega + \pi^*\phi = 0$ and $\int_{S_p^*M} \omega = 0$ for all $p \in M$. There exists a unique form $\omega' = \omega + \alpha \wedge \xi$ such that $d\omega' = D\omega$. Note that $\Psi_{(\omega,\phi)} = \Psi_{(\omega',\phi)}$.

Let $p \in M$ and $U \subset M$ be a contractible neighborhood of p. Let $X \in \mathcal{C}_b(M)$ with $X \subset U$. Fix $\psi \in \Omega^{n-1}(U)$ with $d\psi = \phi$ on U.

Since $X \subset U$, we have

$$[[\partial X]] = \pi_* \operatorname{cnc}(X),$$
$$[\operatorname{cnc}(X)] = \chi(X)[[S_p^*M]] \in H_{n-1,dR}(S^*U) \text{ where } p \in U.$$

It follows that

$$\begin{aligned}
\Psi_{(\omega,\phi)}(X) &= \Psi_{(\omega',\phi)}(X) \\
&= \operatorname{cnc}(X)(\omega') + \int_X \phi \\
&= \operatorname{cnc}(X)(\omega') + \int_X d\psi \\
&= \operatorname{cnc}(X)(\omega' + \pi^*\psi).
\end{aligned}$$

By assumption $d(\omega' + \pi^*\psi) = 0$ on U. Therefore

$$\operatorname{cnc}(X)(\omega' + \pi^*\psi) = \chi(X) \int_{S_p^*M} (\omega' + \pi^*\psi) = \chi(X) \int_{S_p^*M} \omega = 0.$$

By Euler-additivity of $\Psi_{(\omega,\phi)}$ we obtain that $\Psi_{(\omega,\phi)}(X) = 0$ for all $X \in \mathcal{C}_b(M)$, i.e., $(\omega, \phi) \in \ker(\mathbf{\Psi})$.

2) Let us now suppose that $\Psi_{(\omega,\phi)} = 0$. We first want to show that $D\omega = -\pi^*\phi$. By replacing ω with $\omega + \alpha \wedge \xi$ if necessary, we may assume that $D\omega = d\omega$. We set $\eta := i_T D\omega$, where T is the Reeb vector field associated to α. Then $D\omega = \alpha \wedge \eta$.

Given a smooth vector field V on M, there exists a canonical lift V^c on S^*M (i.e., $d\pi(V^c) = V$), which is called *complete lift* (compare [**19**]). If $\Phi^t : M \to M, t \in \mathbb{R}$ denotes the flow generated by V, then $d\Phi^t$ induces a contactomorphism $\tilde{\Phi}^t$ on S^*M by sending (p, P) to $(\Phi^t(p), d\Phi^t_p(P))$. For fixed $(p, P) \in S^*M$, we obtain a curve $t \mapsto (\Phi^t(p), d\Phi^t_p(P))$ in S^*M which starts in (p, P) and which is defined for sufficiently small t. The derivative at $t = 0$ of this curve is the complete lift of V to the point $(p, P) \in S^*M$.

Now let V be real-analytic and $X \in \mathcal{C}_b(M)$. Then, for small t, the conormal cycle of $X_t = \Phi^t(X)$ is the same as the image of the conormal cycle of X under $\tilde{\Phi}^t$ (compare [**11**]), i.e.,

$$\operatorname{cnc}(\Phi_t(X)) = (\tilde{\Phi}_t)_* \operatorname{cnc}(X).$$

Since $\Psi_{(\omega,\phi)} = 0$, we obtain

$$\begin{aligned}
0 &= \left.\frac{d}{dt}\right|_{t=0} \Psi_{(\omega,\phi)}(\Phi^t(X)) \\
&= \left.\frac{d}{dt}\right|_{t=0} \operatorname{cnc}(\Phi_t(X))(\omega) + \left.\frac{d}{dt}\right|_{t=0} \int_{\Phi_t X} \phi \\
&= \left.\frac{d}{dt}\right|_{t=0} (\tilde{\Phi}^t)_* \operatorname{cnc}(X)(\omega) + \int_X \mathcal{L}_V \phi \\
&= \operatorname{cnc}(X) \left(\left.\frac{d}{dt}\right|_{t=0} (\tilde{\Phi}^t)^* \omega \right) + \int_X \mathcal{L}_V \phi \\
&= \operatorname{cnc}(X)(\mathcal{L}_{V^c}\omega) + \int_X \mathcal{L}_V \phi.
\end{aligned}$$

Now suppose that $X \subset U$ for some open contractible set $U \subset M$. As above, $\partial[[X]] = \pi_* \operatorname{cnc}(X)$.

We denote by i_V the contraction of a differential form with V. Using Cartan's formula $\mathcal{L}_V = di_V + i_V d$, we get $\mathcal{L}_V \phi = di_V \phi$ and therefore $\int_X \mathcal{L}_V \phi = \operatorname{cnc}(X)(\pi^* i_V \phi)$.

Using Cartan's formula again we see that

$$\mathcal{L}_{V^c}\omega = di_{V^c}\omega + i_{V^c} d\omega = di_{V^c}\omega + \alpha(V^c)\eta - \alpha \wedge i_{V^c}\eta.$$

Since $\partial \operatorname{cnc}(X) = 0$ and $\operatorname{cnc}(X) \llcorner \alpha = 0$ we deduce that

$$\operatorname{cnc}(X)(\alpha(V^c)\eta + \pi^* i_V \phi) = 0 \tag{2}$$

for all $X \in \mathcal{C}_b(M)$ with $X \subset U$ and all real-analytic vector fields V on M.

By approximation, equation (2) even holds for all smooth vector fields V.

Recall that an $n-1$-dimensional linear subspace E of $T_{(p,P)}S^*M$ is called *Legendrian* if α and $d\alpha$ vanish on E. Equivalently, E is a Lagrangian subspace of the contact plane. We call E *regular* if $d\pi_{(p,P)}\big|_E$ is injective. The set of regular Legendrian subspaces is dense in the space of all Legendrian subspaces.

Let $X \subset \mathcal{C}_b(M)$ of dimension n, with smooth boundary ∂X and such that $X \subset U$. The conormal cycle of X is given by integration over the canonically oriented $n-1$-dimensional manifold $C := \{(x, T_x^o\partial X) : x \in \partial X\} \subset S^*M$.

Fix a point $(p,P) \in S^*M$ with $p \in U$. Let E be an oriented, regular Legendrian subspace of $T_{(p,P)}S^*M$. Since E is regular, we can choose X as above in such a way that $(p,P) \in C$ and $T^o_{(p,P)}C = E$.

Choose a smooth vector field V on M in such a way that $V(p) \notin P$ and a smooth cutoff-function $f \in C^\infty(U)$ with $f(p) = 1$. The definition of the complete lift implies that $\alpha((fV)^c) = (f \circ \pi)\alpha(V^c)$.

It follows that

$$\operatorname{cnc}(X)(f \circ \pi \cdot (\alpha(V^c)\eta + \pi^* i_V \phi)) = 0. \tag{3}$$

Letting the support of f shrink to the point p, we obtain that $(\alpha(V^c)\eta + \pi^* i_V\phi)(E) = 0$. By density of regular Legendrian spaces, this equation even holds for all Legendrian spaces E.

Let $\xi \in T_p^*M$ with $\ker \xi = P$. Then $\alpha|_{(p,P)} = c\pi^*\xi$ for some real $c \neq 0$. Since $\phi|_p \in \Lambda^n T_p^*M$, we have $\xi \wedge \phi = 0$ and therefore $\xi(V)\phi = \xi \wedge i_V\phi$. Taking pull-backs, we obtain $\alpha(V^c) \wedge \pi^*\phi = \alpha \wedge \pi^* i_V \phi$. We multiply both sides by $d\alpha$ and obtain $0 = d\alpha \wedge \alpha \wedge \pi^* i_V\phi$. This implies that the restriction of $\pi^* i_V \phi$ to the contact plane at (p,P) is primitive.

By definition of D, $d\alpha \wedge D\omega = 0$. It follows that the restriction of η to the contact plane is primitive.

Now we use the following lemma, whose proof is given below.

Lemma 1.4. *Let (V,Ω) be a symplectic vector space of dimension $2m$. Let $L : \Lambda^* V^* \to \Lambda^{*+2}V^*$ denote the associated Lefschetz operator (i.e., multiplication by Ω). Let $\beta \in \Lambda^k V^*, k \leq m$ be primitive (i.e., $L^{m-k+1}\beta = 0$). If β vanishes on all isotropic k-dimensional linear subspaces of V then $\beta = 0$.*

Since the restriction of $\alpha(V^c)\eta + \pi^* i_V\phi$ to the contact plane at (p,P) is primitive and vanishes on all Lagrangian subspaces, it has to vanish. Since this is true for all (p,P), we obtain that $\alpha(V^c)\eta + \pi^* i_V \phi$ is vertical, i.e.,

$$\alpha \wedge (\alpha(V^c)\eta + \pi^* i_V\phi)|_{(p,P)} = 0.$$

From $\alpha \wedge \eta = D\omega$ and $\alpha(V^c) \wedge \pi^*\phi = \alpha \wedge \pi^* i_V\phi$, we deduce that

$$\alpha(V^c)(D\omega + \pi^*\phi)_{(p,P)} = 0.$$

By assumption, $V(p) \notin P$, which implies that $\alpha_{(p,P)}(V^c) \neq 0$. Therefore $(D\omega + \pi^*\phi)_{(p,P)} = 0$. Since this is true for all $(p,P) \in S^*M$, we have $D\omega + \pi^*\phi = 0$.

The second statement follows from $\int_{S_p^*M} \omega = \Psi_{(\omega,\phi)}(\{p\}) = 0$. q.e.d.

Proof of Lemma 1.4. Fix a compatible complex structure I on V. Let $e_1, \ldots, e_m, f_1 = Ie_1, \ldots, f_m = Ie_m$ be a symplectic base of V. The dual Lefschetz operator Λ of $\beta \in \Lambda^k V^*$ is given by

$$\Lambda\beta(v_1, \ldots, v_{k-2}) = \sum_{i=1}^m \beta(e_i, f_i, v_1, \ldots, v_{k-2}), \quad v_1, \ldots, v_{k-2} \in V.$$

The condition $L^{m-k+1}\beta = 0$ is equivalent to $\Lambda\beta = 0$. Note that $\Lambda\beta = 0$ implies $\beta = 0$ in the case $k > m$.

Let us prove the statement of the lemma by induction on m, the case $m = 1$ being trivial.

In the case $m > 1$, we denote the linear span of $e_1, f_1, \ldots, e_{m-1}, f_{m-1}$ with the induced symplectic structure by V'.

Let $e_1^*, f_1^*, \ldots, e_m^*, f_m^* \in V^*$ denote the dual base. We can write β uniquely as

$$\beta = \beta_1 + \beta_2 \wedge e_m^* + \beta_3 \wedge f_m^* + \beta_4 \wedge e_m^* \wedge f_m^*$$

with $\beta_1 \in \Lambda^k V'^*, \beta_2, \beta_3 \in \Lambda^{k-1}V'^*, \beta_4 \in \Lambda^{k-2}V'^*$.

The equation $\Lambda\beta = 0$ is equivalent to $\Lambda'\beta_2 = 0$, $\Lambda'\beta_3 = 0$, $\Lambda'\beta_4 = 0$ and $\Lambda'\beta_1 + \beta_4 = 0$, where Λ' denotes the dual Lefschetz operator of V'.

If E' is an isotropic $k-1$-dimensional subspace of V', then $E' \wedge e_m$ and $E' \wedge f_m$ are isotropic k-dimensional subspaces of V. Since β vanishes by assumption on such spaces, the induction hypothesis implies that $\beta_2 = \beta_3 = 0$.

For $i = k-1, \ldots, m-1$, the vectors $e_1, \ldots, e_{k-2}, e_i + e_m, f_i - f_m$ span an isotropic subspace of V. Since β vanishes on it, we get

$$\beta_1(e_1, \ldots, e_{k-2}, e_i, f_i) - \beta_4(e_1, \ldots, e_{k-2}) = 0.$$

Now we sum over $i = k-1, \ldots, m-1$, use $\Lambda'\beta_1 + \beta_4 = 0$ and get

$$(m-k+2)\beta_4(e_1, \ldots, e_{k-2}) = 0.$$

The choice of the symplectic base $e_1, f_1, \ldots, e_{m-1}, f_{m-1}$ of V' being arbitrary, the primitive element β_4 vanishes on $k-2$-dimensional isotropic subspaces of V'. By induction hypothesis $\beta_4 = 0$ and thus $\Lambda'\beta_1 = 0$. Since β vanishes on k-dimensional isotropic subspaces of V', the induction hypothesis implies that $\beta_1 = 0$ and thus $\beta = 0$. q.e.d.

Corollary 1.5. *If $D\omega + \pi^*\phi = 0$, then $r := \int_{S_p^*M} \omega \in \mathbb{R}$ is independent of $p \in M$ and*

$$\Psi_{(\omega,\phi)} = r\chi$$

where χ denotes Euler characteristic.

Proof. Let U be a contractible open subset of M. Write $\phi = d\psi$ with $\psi \in \Omega^{n-1}(U)$. Since $d\pi^*\phi = 0$ and $\alpha \wedge \pi^*\phi = 0$, we have $\pi^*\phi \in \mathcal{J}^n$. Therefore $D\pi^*\psi = d\pi^*\psi = \pi^*\phi$ on U.

From Theorem 1 we deduce that the valuations $\Psi_{(\omega,\phi)}$ and $\Psi_{(\omega+\pi^*\psi,0)}$ agree for subsets of U. From $D(\omega + \pi^*\psi) = 0$ we deduce that $d(\omega + \pi^*\psi + \alpha \wedge \xi) = 0$ for some $\xi \in \Omega^{n-2}(U)$. Since U is connected, the value $\int_{S_p^*M} (\omega + \pi^*\psi + \alpha \wedge \xi) = \langle [\omega + \pi^*\psi + \alpha \wedge \xi], [S_p^*M] \rangle$ is independent of $p \in U$ (by Stokes's theorem). But the last two terms do not contribute to the integral. Hence $r := \int_{S_p^*M} \omega$ is independent of $p \in U$ and, since M is connected, independent of $p \in M$.

Let us prove the second assertion. By Fu's generalization of the Gauss-Bonnet-Chern theorem, χ is a smooth valuation, say $\chi = \Psi_{(\omega',\phi')}$ with $D\omega' + \pi^*\phi' = 0$ and $\int_{S_p^*M} \omega' = 1$ (compare [**10**], 1.5. and 1.8.). From Theorem 1 we deduce that $\Psi_{(\omega,\phi)} = \Psi_{(r\omega',r\phi')} = r\chi$. q.e.d.

Corollary 1.6. *If M is not compact, we can write each smooth valuation Ψ as $\Psi = \Psi_\omega := \Psi_{(\omega,0)}$ for some $\omega \in \Omega^{n-1}(S^*M)$.*

Proof. Suppose that $\Psi = \Psi_{(\omega,\phi)}$. Let $\psi \in \Omega^{n-1}(M)$ with $d\psi = \phi$. Then $d\pi^*\psi = \pi^*\phi \in \mathcal{J}^n$, i.e., $D\pi^*\psi = \pi^*\phi$. Theorem 1 implies that $\Psi_{(\omega,\phi)} = \Psi_{(\omega+\pi*\psi,0)}$. q.e.d.

Let G be any group acting by $\mathcal{C}$-diffeomorphisms on M. There is an induced action on $\mathcal{V}^\infty(M)$, given by $(g\Psi)(X) := \Psi(g^{-1}(X))$. The subspace of G-invariant valuations is denoted by $\mathcal{V}^\infty(M)^G$.

Also, there is an induced G-action on S^*M, given by $(g,(p,P)) \mapsto (gp, d_pg(P))$, which leaves the contact distribution invariant. Given $g \in G$, we set $\varepsilon_g = 1$ if g preserves the orientation of M and $\varepsilon = -1$ otherwise. We let G act on $\Omega^*(S^*M)$ by $(g,\omega) \mapsto \varepsilon_g g^*\omega$. Then G commutes with d_Q and D. Similarly, we let G act on $\Omega^*(M)$ by $(g,\phi) \mapsto \varepsilon_g^*\phi$.

Proposition 1.7. *Let $\Psi_{(\omega,\phi)} \in \mathcal{V}^\infty(M)^G$. Then $D\omega + \pi^*\phi$ is G-invariant.*

Proof. If $\Psi_{(\omega,\phi)}$ is G-invariant, then $\Psi_{(\omega,\phi)}(gX) = \Psi_{(\omega,\phi)}(X)$ for all $X \in \mathcal{C}_b(M)$ and all $g \in G$.

Since $\operatorname{cnc}(gX) = \varepsilon_g g_* \operatorname{cnc}(X)$ and $[[gX]] = \varepsilon_g g_*[[X]]$, we obtain $\Psi_{(\omega,\phi)}(gX) = \varepsilon_g \Psi_{(g^*\omega,g^*\phi)}(X)$. It follows that $\Psi_{(g^*\omega,g^*\phi)} = \varepsilon_g \Psi_{(\omega,\phi)}$. From Theorem 1 we deduce that

$$D(\varepsilon_g g^*\omega - \omega) + \pi^*(\varepsilon_g g^*\phi - \phi) = 0$$

for all $g \in G$, i.e., $D\omega + \pi^*\phi$ is G-invariant. q.e.d.

We remark that the G-invariance of Ψ_ω does not imply the G-invariance of ω. For instance, let $\omega := \pi^* x_1 dx_2 \dots dx_n$ on $S^*\mathbb{R}^n$. Then

$\Psi_\omega(X) = \mathrm{vol}_n(X)$ for all $X \in \mathcal{C}_b(\mathbb{R}^n)$. In particular, Ψ_ω is invariant under Euclidean motions. However, ω is not invariant.

2. Operations on valuations

2.1. Euler–Verdier involution. The Euler–Verdier involution of smooth valuations was introduced by Alesker [**7**]. It can easily be described in terms of the contact geometry. The map s which changes the orientation of $(p,P) \in S^*M$ is an involution on S^*M preserving the contact structure. It induces an involution s^* on $\Omega^{n-1}/\mathcal{I}^{n-1}$.

Theorem 2.1. *There exists a unique involution $\sigma : \mathcal{V}^\infty(M) \to \mathcal{V}^\infty(M)$ such that the following diagram commutes:*

$$\begin{array}{ccc} \Omega^{n-1}(S^*M) \oplus \Omega^n(M) & \overset{((-1)^n s^*, (-1)^n id)}{\longrightarrow} & \Omega^{n-1}(S^*M) \oplus \Omega^n(M) \\ \mathbf{\Psi}\downarrow & & \mathbf{\Psi}\downarrow \\ \mathcal{V}^\infty(M) & \overset{\sigma}{\longrightarrow} & \mathcal{V}^\infty(M) \end{array}$$

Proof. If σ exists, then $\sigma\Psi_{(\omega,\phi)} = \Psi_{((-1)^n s^*\omega,(-1)^n\phi)}$ is unique.

Let us show that this defines σ. If $\Psi_{(\omega,\phi)} = 0$, then by Theorem 1 $D\omega + \pi^*\phi = 0$ and $\int_{S_p^*M} \omega = 0$. It follows that $D(-1)^n s^*\omega + \pi^*(-1)^n\phi = 0$ and, since $s_*[[S_p^*M]] = \pm[[S_p^*M]]$, $\int_{S_p^*M} s^*\omega = \pm\int_{S_p^*M}\omega = 0$. Therefore $\Psi_{((-1)^n s^*\omega,(-1)^n\phi)} = 0$. q.e.d.

2.2. Signature operator and Laplacian. From now on, we suppose that $(M, \langle\cdot,\cdot\rangle)$ is a Riemannian manifold. It will be more convenient to work with SM instead of S^*M. On SM, there exists a canonical global 1-form α defined by $\alpha|_{(p,v)}(X) = \langle v, \pi_* X\rangle$ for all $X \in T_{(p,v)}SM$. Here $\pi : SM \to M$ is the natural projection map. The Riemannian metric induces a contactomorphism between S^*M and SM. If $X \in \mathcal{C}_b(M)$, then the image of the conormal cycle of X under this contactomorphism is the normal cycle $\mathrm{nc}(X)$, an integral Legendrian $n-1$-cycle on SM.

Given forms $\omega \in \Omega^{n-1}(SM)$ and $\phi \in \Omega^n(M)$, the valuation $\Psi_{(\omega,\phi)}$ defined by $X \mapsto \mathrm{nc}(X)(\omega) + \int_X \phi$ is smooth. Theorem 1 now reads as follows.

Theorem 2.2. *Let (M,g) be a real-analytic Riemannian manifold of dimension n and $\mathbf{\Psi} : \Omega^{n-1}(SM) \oplus \Omega^n(M) \to \mathcal{V}^\infty(M), (\omega,\phi) \mapsto \Psi_{(\omega,\phi)}$. Then $(\omega,\phi) \in \ker(\mathbf{\Psi})$ if and only if $D\omega + \pi^*\phi = 0$ and $\int_{S_pM}\omega = 0$ for all $p \in M$.*

Proof. Immediate from Theorem 1. q.e.d.

The metric on M induces a natural metric on SM, called the *Sasaki metric* [**19**]. We get an induced metric on differential forms on SM and a duality operator $* : \Omega^k(SM) \to \Omega^{2n-1-k}(SM)$ such that $\omega \wedge *\xi = \langle\omega,\xi\rangle \alpha \wedge d\alpha^{n-1}$.

Following Rumin [**17**], we denote by $\delta_Q := (-1)^k * d_Q *$ $(k \neq n)$ and $D^* := (-1)^n * D*$ the dual operators of d_Q and D.

Theorem 2.3. *There exists a unique operator $\mathcal{S} : \mathcal{V}^\infty(M) \to \mathcal{V}^\infty(M)$, called signature operator, such that the following diagram commutes*

$$\begin{array}{ccc} \Omega^{n-1}(SM) \oplus \Omega^n(M) & \xrightarrow{(*D+*\circ\pi^*,0)} & \Omega^{n-1}(SM) \oplus \Omega^n(M) \\ \mathbf{\Psi} \downarrow & & \mathbf{\Psi} \downarrow \\ \mathcal{V}^\infty(M) & \xrightarrow{\mathcal{S}} & \mathcal{V}^\infty(M) \end{array}$$

Proof. If $\mathcal{S}$ exists, it is uniquely given by $\mathcal{S}\Psi_{(\omega,\phi)} = \Psi_{*D\omega+*\pi^*\phi}$ for all $\omega \in \Omega^{n-1}(M)$ and $\phi \in \Omega^n(M)$.

This is a well-defined operator. Indeed, if $(\omega, \phi) \in \ker \mathbf{\Psi}$, then $*(D\omega + \pi^*\phi) = 0$ by Theorem 2.2. q.e.d.

The Rumin-Laplace operator Δ_Q acts on $\Omega^k/\mathcal{I}^k$ for $0 \leq k \leq n-1$ and on $\mathcal{J}^k$ for $n \leq k \leq 2n-1$ by

$$\Delta_Q = \begin{cases} (n-k-1)d_Q\delta_Q + (n-k)\delta_Q d_Q & 0 \leq k \leq n-2 \\ (d_Q\delta_Q)^2 + D^*D & k = n-1 \\ DD^* + (\delta_Q d_Q)^2 & k = n \\ (n-k)d_Q\delta_Q + (n-k-1)\delta_Q d_Q & n+1 \leq 2n-1. \end{cases}$$

We set $\Delta := (-1)^n\mathcal{S}^2 : \mathcal{V}^\infty(M) \to \mathcal{V}^\infty(M)$ and call Δ the *Laplacian acting on (smooth) valuations.*

Proposition 2.4. *The following diagram commutes:*

$$\begin{array}{ccc} \Omega^{n-1}(SM) \oplus \Omega^n(M) & \xrightarrow{(\Delta_Q+D^*\circ\pi^*,0)} & \Omega^{n-1}(SM) \oplus \Omega^n(M) \\ \mathbf{\Psi} \downarrow & & \mathbf{\Psi} \downarrow \\ \mathcal{V}^\infty(M) & \xrightarrow{\Delta} & \mathcal{V}^\infty(M) \end{array}$$

Proof. By Theorem 2.2, $\Psi_{(d_Q\delta_Q)^2\omega} = 0$. Therefore,

$$\mathcal{S}^2\Psi_{(\omega,\phi)} = \mathcal{S}\Psi_{*D\omega+*\pi^*\phi} = \Psi_{*D*D\omega+*D*\pi^*\phi} = (-1)^n\Psi_{\Delta_Q\omega+D^*\pi^*\phi}.$$

q.e.d.

2.3. Derivation operator. Let T be the Reeb vector field on SM, i.e., $\alpha(T) = 1$ and $\mathcal{L}_T\alpha = 0$, where $\mathcal{L}_T$ is the Lie derivative.

Theorem 2.5. *There exists a unique operator $\mathfrak{L} : \Omega^{sm}(M) \to \Omega^{sm}(M)$, called derivation operator, such that the following diagram commutes:*

$$\begin{array}{ccc} \Omega^{n-1}(SM) \oplus \Omega^n(M) & \xrightarrow{(\mathcal{L}_T+i_T\pi^*,0)} & \Omega^{n-1}(SM) \oplus \Omega^n(M) \\ \mathbf{\Psi} \downarrow & & \mathbf{\Psi} \downarrow \\ \mathcal{V}^\infty(M) & \xrightarrow{\mathfrak{L}} & \mathcal{V}^\infty(M) \end{array}$$

Proof. Note first that D and $\mathcal{L}_T$ commute. For, if $\omega \in \Omega^{n-1}$, then $D\omega = d(\omega + \alpha \wedge \xi) = \alpha \wedge \eta$ and, since $\mathcal{L}_T$ and d commute, we get $d(\mathcal{L}_T\omega + \alpha \wedge \mathcal{L}_T\xi) = \alpha \wedge \mathcal{L}_T\eta$. This shows that $D\mathcal{L}_T\omega = \mathcal{L}_T D\omega$.

If the operator $\mathfrak{L}$ exists, then $\mathfrak{L}\Psi_{(\omega,\phi)} = \Psi_{\mathcal{L}_T\omega + i_T\pi^*\phi}$. We have to show that this is a well-defined operator.

If $\Psi_{(\omega,\phi)} = 0$, then $D\omega + \pi^*\phi = 0$ by Theorem 2.2. Let $\xi \in \Omega^{n-2}(SM)$ be such that $D\omega = d(\omega + \alpha \wedge \xi)$. Then

$$\begin{aligned}
\mathcal{L}_T\omega + i_T\pi^*\phi &= \mathcal{L}_T(\omega + \alpha \wedge \xi) - \mathcal{L}_T(\alpha \wedge \xi) + i_T\pi^*\phi \\
&= i_T D\omega + d i_T(\omega + \alpha \wedge \xi) - \alpha \wedge \mathcal{L}_T\xi + i_T\pi^*\phi \\
&= d i_T(\omega + \alpha \wedge \xi) - \alpha \wedge \mathcal{L}_T\xi
\end{aligned}$$

and thus $\Psi_{\mathcal{L}_T\omega + i_T\pi^*\phi} = 0$. q.e.d.

3. Translation invariant valuations on Euclidean spaces

Let $\mathrm{Val}(V)$ be the space of translation invariant continuous convex valuations on V. Equipped with the topology of uniform convergence on compact subsets of $\mathcal{K}(V)$, $\mathrm{Val}(V)$ is a Fréchet space.

A valuation has degree k, if $\Psi(tK) = t^k\Psi(K)$ for all $K \in \mathcal{K}(V)$ and all $t > 0$. Ψ is called *even* if $\Psi(-K) = \Psi(K)$ and *odd* if $\Psi(-K) = -\Psi(K)$ for all $K \in \mathcal{K}(V)$.

McMullen [**16**] proved that there is a decomposition

$$\mathrm{Val}(V) = \bigoplus_{k=0}^{n} \mathrm{Val}_k(V). \tag{4}$$

Furthermore, each $\mathrm{Val}_k(V)$ splits as $\mathrm{Val}_k(V) = \mathrm{Val}_k^{ev}(V) \oplus \mathrm{Val}_k^{odd}(V)$, where $\mathrm{Val}_k^{ev}(V)$ and $\mathrm{Val}_k^{odd}(V)$ denote even and odd valuations of degree k. The natural $GL(V)$-representation in $\mathrm{Val}(V)$ preserves degree and parity.

Theorem 3.1 (Alesker's Irreducibility Theorem, [**2**])**.** *The natural $GL(V)$-representations in $\mathrm{Val}_k^{ev}(V)$ and $\mathrm{Val}_k^{odd}(V)$ are irreducible.*

We can speak of smooth valuations in the representation theoretic sense (compare [**4**]) and in the sense of Definition 1.3, and it is not a priori clear that the two notions are the same. However, using his irreducibility theorem and the Casselmann-Wallach-theorem, Alesker showed the following theorem.

Theorem 3.2 (Alesker, [**6**])**.** *Let Ψ be a translation invariant, continuous convex valuation. Then Ψ is smooth with respect to the natural $GL(V)$-action if and only if it is smooth in the sense of Definition* 1.3.

As we have seen, the translation invariance of the valuation Ψ_ω does not imply the translation invariance of ω. The next theorem describes the relation between smooth translation invariant valuations and translation invariant differential forms.

In the following, the subscript $k, n-k$ will denote the component of bidegree $(k, n-k)$ (w.r.t. the product structure $SV = V \times S(V)$).

Theorem 3.3.

1) *For* $1 \leq k \leq n-1$, *there is an injective map*

$$\operatorname{Val}_k^{sm}(V) \longrightarrow (\operatorname{im} D)_{k,n-k}^V = (\ker d)_{k,n-k}^V \cap \mathcal{J}^n(SV).$$

2) *For* $2 \leq k \leq n-1$, *this map is also surjective, i.e., an isomorphism.*
3) *For* $k = 1$, *the above map induces an isomorphism*

$$\operatorname{Val}_1^{sm}(V) \xrightarrow{\cong} \left\{ f \wedge \alpha \wedge \mu_{S(V)}, f \in C^\infty(S(V)), \int_{S(V)} y f(y) d\mu_{S(V)}(y) = 0 \right\}. \tag{5}$$

4) *Let* $0 \leq k \leq n-1$ *and* $l := \binom{n}{k}$. *Then each* $\Psi \in \operatorname{Val}_k^{sm}(V)$ *can be written in the form*

$$\Psi = \Psi_\omega, \omega = \sum_{i=1}^l \kappa_i \wedge \xi_i \in \Omega^{n-1}(SV)_{k,n-k-1}^V$$

where $\kappa_1, \ldots, \kappa_l$ *is a basis of translation invariant* k*-forms on* V, $\xi_i \in \Omega^{n-k-1}(S(V))$ *are coclosed and* $D\omega = d\omega$. *This representation is unique if* $k < n-1$. *In the case* $k = n-1$, ξ_i *are functions on* $S(V)$ *and the representation is unique under the assumption* $\int_{S(V)} \xi_i d\mu = 0$.

Proof.

1) By Corollary 1.6, each $\Psi \in \operatorname{Val}_k^{sm}(V)$ can be written in the form $\Psi = \Psi_\omega$ with $\omega \in \Omega^{n-1}(SV)$. Using Theorem 2.2, we get a map $\operatorname{Val}_k^{sm}(V) \to \operatorname{im} D, \Psi_\omega \mapsto D\omega$. The image is contained in $(\operatorname{im} D)^V$ by Proposition 1.7.

   We claim that the bidegree of $D\omega$ is $(k, n-k)$. For $t > 0$, we let $m_t(x,y) := (tx, y)$. Since Ψ is of degree k, we have $\Psi(tK) = t^k \Psi(K)$. But $\Psi(tK) = \operatorname{nc}(tK)(\omega) = (m_t)_* \operatorname{nc}(K)(\omega) = \operatorname{nc}(K)(m_t^*\omega) = \Psi_{m_t^*\omega}(K)$. From Theorem 2.2 we infer that $m_t^* D\omega - t^k D\omega = D(m_t^*\omega - t^k\omega) = 0$ for $t \geq 0$. This implies that the bidegree of $D\omega$ is $(k, n-k)$.

   Let us show injectivity. Suppose $\Psi = \Psi_\omega \in \operatorname{Val}_k^{sm}(V)$ such that $D\omega = 0$. Since Ψ is of degree $k > 0$, $0 = \Psi(\{x\}) = \int_{S_xV} \omega$ for all $x \in V$. Theorem 2.2 implies that $\Psi = 0$.

   The equality $(\operatorname{im} D)_{k,n-k}^V = (\ker d)_{k,n-k}^V \cap \mathcal{J}^n(SV)$ follows at once from the Rumin isomorphism (1).

2) Suppose $k \geq 2$. To show surjectivity, we let $\psi \in (\operatorname{im} D)^V_{k,n-k}$. Let $\kappa_1, \ldots, \kappa_l$ be a basis of translation invariant k-forms on V. Then $d\kappa_i = 0$ for $i = 1, \ldots, l$.

   We can write ψ uniquely as

$$\psi = \sum_{i=1}^{l} \kappa_i \wedge \tau_i,$$

   where $\tau_i, i = 1, \ldots, l$ are $n-k$-forms on $S(V)$.

   Now $0 = d\psi = (-1)^k \sum_{i=1}^l \kappa_i \wedge d\tau_i$, since $\psi \in \operatorname{im} D = \ker d_Q$. It follows that $d\tau_i = 0$ for all i. Using $H^{n-k}_{dR}(S(V)) = 0$ and the Hodge-de Rham theorem, we can write $\tau_i = (-1)^k d\xi_i$ for some coclosed $n-k-1$-forms ξ_i on $S(V)$.

   We set $\omega := \sum_{i=1}^l \kappa_i \wedge \xi_i$. Then $D\omega = d\omega = \psi$. Since ω is translation invariant, the same is true for the valuation $\Psi := \Psi_\omega$. Moreover, since ω is of bidegree $(k, n-k-1)$, Ψ is of degree k.
3) Any form $\psi \in (\operatorname{im} D)^V_{1,n-1}$ can be written as $\psi = \alpha \wedge \xi$ with an $n-1$-form ξ on $S(V)$. Since we can write $\xi = f\mu_{S(V)}$ for some $f \in C^\infty(S(V))$, we get $\psi = f \wedge \alpha \wedge \mu_{S(V)}$. On the other hand, any such form is closed and thus belongs to $(\operatorname{im} D)^V_{1,n-1}$.

   Now suppose that there exists a valuation $\Psi = \Psi_\omega \in \mathrm{Val}_1^{sm}(V)$ such that $D\omega = \psi = f \wedge \alpha \wedge \mu_{S(V)}$.

   Use coordinates $(x_1, ..., x_n)$ on V and induced coordinates (x, y) on $V \times V$.

   Set $\omega' := fu \wedge \mu_{S(V)} \in \Omega^{n-1}(SV)$, where $u : V \times V \to \mathbb{R}, (x,y) \mapsto \sum_{i=1}^n x_i y_i$ is the scalar product of V. Then $D\omega' = d\omega' = \psi$ and Corollary 1.5 implies that $\Psi_{\omega'} - \Psi = r\chi$ for some $r \in \mathbb{R}$. Since Ψ is of degree 1, $\Psi(\{x\}) = 0$ for all $x = (x_1, \ldots, x_n) \in V$. But

$$\Psi_{\omega'}(\{x\}) = \int_{S_x V} \omega' = \sum_{i=1}^n x_i \int_{S(V)} y_i f(y) \mu_{S(V)}$$

   is independent of x if and only if $\int_{S(V)} y f(y) \mu_{S(V)} = 0$.

   To show the other direction, suppose that $\psi = f \wedge \alpha \wedge \mu_{S(V)}$ with $\int_{S(V)} y f(y) d\mu_{S(V)} = 0$.

   Then we find coclosed $n-2$-forms ξ_i on $S(V)$ with $d\xi_i = -y_i f(y) \mu_{S(V)}$. Setting $\omega := \sum_{i=1}^n dx_i \wedge \xi_i$ we get $D\omega = d\omega = \psi$. The valuation Ψ_ω belongs to $\mathrm{Val}_1^{sm}(V)$, which finishes the proof of Equation (5).

   We note that, by Corollary 1.7, a valuation $\Psi \in \mathrm{Val}_1^{sm}(V)$ is even (odd) if and only if the function f on $S(V)$ is even (odd).
4) Suppose first that $1 \leq k \leq n-1$. As we have seen in the proof of (2) and (3), each $\Psi \in \mathrm{Val}_k^{sm}(V)$ can be written as $\Psi = \Psi_\omega$ with $\omega = \sum_{i=1}^l \kappa_i \wedge \xi_i$, $\delta\xi_i = 0$, $D\omega = d\omega$. It remains to prove uniqueness.

Suppose that $\omega' = \sum_{i=1}^l \kappa_i \wedge \xi_i'$ satisfies the same conditions and $\Psi = \Psi_{\omega'}$. By Theorem 2.2, $d(\omega - \omega') = D(\omega - \omega') = 0$. Therefore $d(\xi_i - \xi_i') = 0$ for all i. By assumption $\delta(\xi_i - \xi_i') = 0$, i.e., $\xi_i - \xi_i'$ is harmonic. If $k < n-1$, $H^{n-k-1}(S(V)) = 0$ and thus there are no non-zero harmonic $n-k-1$-forms on $S(V)$, which implies that $\xi_i = \xi_i'$ and thus $\omega = \omega'$.

If $k = n-1$, then $\xi_i - \xi_i'$ is a constant. The additional assumption $\int_{S(V)} \xi_i d\mu_{S(V)} = \int_{S(V)} \xi_i' d\mu_{S(V)} = 0$ implies that this constant is 0 and hence $\omega = \omega'$.

Let us consider the case $k = 0$. Since $\mathrm{Val}_0^{sm}(V)$ is generated by $\chi = \Psi_\omega$, where ω is the volume form on $S(V)$ (which is co-closed) the existence part of the statement holds. Moreover, each harmonic $n-1$-form on $S(V)$ is a multiple of the volume form, which implies the uniqueness part.

q.e.d.

We end this section by showing that the derivation operator $\mathfrak{L}$ corresponds to one of Alesker's operators (which is denoted by Λ in [**3**]).

Lemma 3.4. *Let Ψ be a smooth valuation on V. Then for all $K \in \mathcal{K}(V)$*

$$\mathfrak{L}\Psi(K) = \left.\frac{d}{dt}\right|_{t=0} \Psi(K + tB).$$

Proof. By Corollary 1.6, there exists a form ω such that $\Psi = \Psi_\omega$.

Let exp be the exponential flow on SV. It is generated by the Reeb vector field T. The normal cycle of $K + tB$ is given by $(\exp_t)_* \operatorname{nc}(K)$ (compare [**8**] for the relation between the Minkowski sum and normal cycles). Therefore

$$\left.\frac{d}{dt}\right|_{t=0} \Psi(K + tB) = \left.\frac{d}{dt}\right|_{t=0} \operatorname{nc}(K)(\exp_t^* \omega) = \operatorname{nc}(K)(\mathcal{L}_T \omega) = \mathfrak{L}\Psi(K).$$

q.e.d.

4. Hard Lefschetz theorem

Proof of Theorem 2. The statement in the case $k = n$ follows easily from Steiner's tube formula.

Let us suppose that $k \leq n-1$. Let $M := V \times (V \setminus \{0\})$. M is in a natural way a (non-compact) Kähler manifold. We denote by I the (almost) complex structure on M.

Given an orthogonal basis on V, we get coordinates $(x_1, \ldots, x_n)$ on V and induced coordinates $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ on M. The vector fields $\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i}, i = 1, \ldots, n$ span TM and $I\left(\frac{\partial}{\partial x_i}\right) = \frac{\partial}{\partial y_i}, I\left(\frac{\partial}{\partial y_i}\right) = -\frac{\partial}{\partial x_i}$. The induced operator on differential forms satisfies $I(dx_i) = -dy_i, I(dy_i) = dx_i$.

The canonical 1-form on M is denoted by $\tilde{\alpha} = \sum_{i=1}^n y_i dx_i$. The Kähler form on M is $\theta := -d\tilde{\alpha}$. Let $\tilde{T}$ be the Hamiltonian vector field with Hamiltonian function $(x,y) \mapsto \frac{1}{2}\|y\|^2$ on M (in coordinates $\tilde{T} = \sum_{i=1}^n y_i \frac{\partial}{\partial x_i}$).

Several operators act on the space of differential forms on M. Besides the differential operators d and $d^I := I^{-1} \circ d \circ I$, their duals $\delta = - * \circ d \circ *$ and $\delta^I = - * \circ d^I \circ *$, we have the (linear) Lefschetz operator L (multiplication by the Kähler form θ) and its dual Λ. Moreover, the Hodge stars on the factors of $M = V \times (V \setminus \{0\})$ induce operators $*_1$ and $*_2$. Note that $*_1 \circ *_2 = *_2 \circ *_1 = (-1)^{l(n-k)} *$ on (k,l)-forms.

In the following, $C_{n,k,\ldots}$ will denote a non-zero real constant which depends on $n, k, \ldots$ and can change its value from line to line.

Proposition 4.1. *Let $\tilde{\omega}$ be a translation invariant (k,l)-form on M. Then*

1) $d *_1 \tilde{\omega} = (-1)^n *_1 d\tilde{\omega}$,
2) $d^I *_2 \omega = *_2 d^I \omega$,
3)
$$\mathcal{L}_{\tilde{T}} \tilde{\omega} = (-1)^{n-k} *_1^{-1} \circ L \circ *_1 \tilde{\omega},$$
4)
$$d^I \tilde{\omega} = C_{n,k,l} \left(*_1^{-1} \Lambda *_1 d\tilde{\omega} - d *_1^{-1} \Lambda *_1 \tilde{\omega} \right), \tag{6}$$
5) *if $L d\tilde{\omega} = 0$ and $\delta \tilde{\omega} = 0$ then*
$$L \Delta \tilde{\omega} = C_{n,k,l} d *_1^{-1} \Lambda *_1 d\tilde{\omega}. \tag{7}$$

Proof. Write $\tilde{\omega} = \sum_{I,J} f_{IJ} dx_I dy_J$, where I ranges over all ordered k-tuples and J ranges over all ordered l-tuples and where f_{IJ} only depends on y but not on x.

1)
$$\begin{aligned} d *_1 \tilde{\omega} &= \sum_{I,J,i} \frac{\partial f_{IJ}}{\partial y_i} dy_i \wedge (*_1 dx_I) \wedge dy_J \\ &= (-1)^{n-k} \sum_{I,J,i} \frac{\partial f_{IJ}}{\partial y_i} (*_1 dx_I) \wedge dy_i \wedge dy_J \\ *_1 d\tilde{\omega} &= *_1 \sum_{I,J,i} \frac{\partial f_{IJ}}{\partial y_i} dy_i \wedge dx_I \wedge dy_J \\ &= (-1)^k \sum_{I,J,i} \frac{\partial f_{IJ}}{\partial y_i} (*_1 dx_I) \wedge dy_i \wedge dy_J. \end{aligned}$$

2)
$$\begin{aligned} d^I *_2 \tilde\omega &= I^{-1} dI *_2 \tilde\omega \\ &= I^{-1} dI \sum_{I,J} f_{I,J} dx_I \wedge *_2 dy_J \\ &= (-1)^k I^{-1} d \sum_{I,J} f_{I,J} dy_I \wedge *_1 dx_J \\ &= (-1)^k I^{-1} \sum_{I,J,i} \frac{\partial f_{I,J}}{\partial y_i} dy_i \wedge dy_I \wedge *_1 dx_J \\ &= -\sum_{I,J,i} \frac{\partial f_{I,J}}{\partial y_i} dx_i \wedge dx_I \wedge *_2 dy_J. \end{aligned}$$

$$\begin{aligned} *_2 d^I \tilde\omega &= *_2 I^{-1} dI \tilde\omega \\ &= (-1)^k *_2 I^{-1} d \sum_{I,J} f_{IJ} dy_I \wedge dx_J \\ &= (-1)^k *_2 I^{-1} \sum_{I,J,i} \frac{\partial f_{IJ}}{\partial y_i} dy_i \wedge dy_I \wedge dx_J \\ &= -\sum_{I,J,i} \frac{\partial f_{IJ}}{\partial y_i} dx_i \wedge dx_I \wedge *_2 dy_J. \end{aligned}$$

3) Since $\mathcal{L}_{\tilde T} f_{IJ} = 0$, it suffices to show the equation in the case $\tilde\omega = dx_I \wedge dy_J$, which is an easy computation.
4) We note that $*_1^2 = (-1)^{k(n-k)}$ and $*_2^2 = (-1)^{l(n-l)}$ on (k,l)-forms. Using the Kähler identity $[\Lambda, d] = -\delta^I$ we compute
$$\begin{aligned} d^I \tilde\omega &= C_{n,k,l} d^I *_2 *_2 \tilde\omega \\ &= C_{n,k,l} *_2 d^I *_2 \tilde\omega \\ &= C_{n,k,l} *_1^{-1} \delta^I *_1 \omega \\ &= C_{n,k,l} \left(*_1^{-1} \Lambda d *_1 \tilde\omega - *_1^{-1} d \Lambda *_1 \tilde\omega \right) \\ &= C_{n,k,l} \left(*_1^{-1} \Lambda *_1 d\tilde\omega - d *_1^{-1} \Lambda *_1 \tilde\omega \right). \end{aligned}$$
5) Using the Kähler identity $[L, \delta] = d^I$ we obtain
$$L\Delta\tilde\omega = \Delta L \tilde\omega = d\delta L\tilde\omega = -d[L,\delta]\tilde\omega = -dd^I\tilde\omega$$
and the result follows from (4).

q.e.d.

Lemma 4.2. *Let ϕ be a differential form on SV. Let $p : M \to SV, (x,y) \mapsto \left(x, \frac{y}{\|y\|}\right)$ denote radial projection.*

1) *If $\alpha \wedge \phi = d\alpha \wedge \phi = 0$, then $Lp^*\phi = 0$, where L is the Lefschetz operator of M.*

2) *For* $k = 0, 1, \ldots$

$$\mathcal{L}_{\tilde{T}}^{k+1} p^*\phi = \|y\|^{k+1} p^* \mathcal{L}_T^{k+1} \phi + d(\|y\|^{k+1}) \wedge p^* i_T \mathcal{L}_T^k \phi.$$

In particular, if $\mathcal{L}_T^k \phi = 0$, *then* $\mathcal{L}_{\tilde{T}}^{k+1} p^* \phi = 0$.

3) $\delta p^* \phi = \|y\|^{-2} p^* \delta\phi$.
4) $\Delta p^* \phi = d\|y\|^{-2} \wedge p^* \delta\phi + \|y\|^{-2} p^* \Delta\phi$.

Proof.

1) The canonical 1-form on M is given by $\tilde{\alpha} = \|y\| p^* \alpha$. Therefore $Lp^*\phi = -d\tilde{\alpha} \wedge p^*\phi = -d\|y\| \wedge p^*(\alpha \wedge \phi) - \|y\| \wedge p^*(d\alpha \wedge \phi) = 0$.
2) It is easily checked that $i_{\tilde{T}} p^* \phi = \|y\| p^* i_T \phi$. The claim then follows by induction using Cartan's formula.
3) Straightforward computation.
4) Follows easily from (3).

q.e.d.

Lemma 4.3. *Let* $\tilde{\omega}$ *be a translation invariant* (k,l)*-form on* M. *For* $t > 0$ *let* $m_t : M \to M, (x,y) \mapsto (x, ty)$. *Suppose that* $m_t^* \tilde{\omega} = t^d \tilde{\omega}$, *i.e.,* $\tilde{\omega}$ *is* d*-homogeneous. If* $\Delta\tilde{\omega} = 0$ *and* $(d-l)(d-l+n-2)$ *is not in the spectrum of* S^{n-1}, *then* $\tilde{\omega} = 0$.

Proof. Write $\tilde{\omega} = \sum_{I,J} f_{IJ} dx_I \wedge dy_J$. Then $\tilde{\omega}$ is d-homogeneous if and only if each function f_{IJ} is $d-l$-homogeneous and $\Delta\tilde{\omega} = 0$ if and only if $\Delta f_{IJ} = 0$ (compare [**15**], 2.1.27). These two equations imply that the restriction of f_{IJ} to S^{n-1} is an eigenfunction with eigenvalue $(d-l)(d-l+n-2)$. q.e.d.

Note that d and $*_1$ preserve the degree of homogeneity, while δ and Δ decrease the degree of homogeneity by 2. If a (k,l)-form $\tilde{\omega}$ is d-homogeneous, then $I\tilde{\omega}$ is $d+k-l$-homogeneous and $*_2\tilde{\omega}$ is $n+d-2l$-homogeneous.

Injectivity of $\mathfrak{L}^{2k-n}$

Let $\Psi \in \mathrm{Val}_k^{sm}(V)$ such that $\mathfrak{L}^{2k-n}\Psi = 0$. Using Theorem 3.3, (4), we can write $\Psi = \Psi_\omega$ where ω is a $(k, n-k-1)$-form on SV with $d\omega = D\omega$ and $\delta\omega = 0$. From Lemma 4.2 we deduce that $Lp^*d\omega = 0$ and $\delta p^*\omega = 0$. Since $0 = \mathfrak{L}^{2k-n}\Psi_\omega = \Psi_{\mathcal{L}_T^{2k-n}\omega}$, Theorem 2.2 implies $D\mathcal{L}_T^{2k-n}\omega = 0$. Since $\mathcal{L}_T$ and D commute (compare the proof of Theorem 2.5), we get $\mathcal{L}_T^{2k-n} d\omega = 0$. By Lemma 4.2, $\mathcal{L}_{\tilde{T}}^{2k-n+1} p^* d\omega = 0$ and thus $L^{2k-n+1} *_1 p^* d\omega = 0$. This means that $*_1 p^* d\omega$ is a primitive $2n-2k$-form, i.e., $\Lambda *_1 p^* d\omega = 0$. Proposition 4.1, (5) implies that $L\Delta p^*\omega = 0$. Since L is bijective on $n-1$-forms on M, we obtain that $\Delta p^*\omega = 0$.

We apply Lemma 4.3 with $d = 0$ and $l = n-k-1$. If $k < n-1$, then $-l(n-l-2) = (k+1-n)(k-1) < 0$. Therefore $p^*\omega = 0$, which implies that $\omega = 0$ and thus $\Psi = 0$.

If $k = n-1$, then each coefficient of $p^*\omega$ is 0-homogeneous and restricts to a harmonic function on the sphere, i.e., is constant. This implies that $d\omega = 0$ and thus $\Psi = 0$ by Theorem 2.2.

Surjectivity of $\mathfrak{L}^{2k-n}$

Let $\Psi' \in \mathrm{Val}_{n-k}^{sm}(V)$ be given. Let us write $\Psi' = \Psi_{\omega'}$ with a translation invariant $(n-k, k-1)$-form ω'. We look for $\Psi \in \mathrm{Val}_k^{sm}(V)$ with $\mathfrak{L}^{2k-n}\Psi = \Psi'$. Using reverse induction on k, we may assume that $\mathfrak{L}\Psi' = 0$, which, by Theorem 2.2, implies that $\mathcal{L}_T D\omega' = D\mathcal{L}_T\omega' = 0$.

Set $\tilde{\beta} := \frac{1}{2}d\|y\|^2 = \sum_{i=1}^n y_i dy_i$ and

$$\psi' := d\|y\|^{2k-n+1} \wedge p^* i_T D\omega'.$$

Claim 1. There exists a unique translation invariant $(k+1, n-k-1)$-form ψ on M such that $\mathcal{L}_{\tilde{T}}^{2k-n+2}\psi = \psi'$. Moreover, $d^I\psi = 0$, $\tilde{\alpha} \wedge \psi = 0$ and $L\psi = \Lambda\psi = 0$.

Indeed, by Proposition 4.1 (3), this equation is equivalent to

$$L^{2k-n+2} *_1 \psi = *_1\psi',$$

the latter being a $2k+2$-form on M. Since $L^{2k-n+2} : \Omega^{2n-2k-2}(M) \to \Omega^{2k+2}(M)$ is bijective by the Lefschetz theorem on M, the existence of ψ follows.

More explicitly, $\psi = C_{n,k} I\psi'$. Indeed, using Lemma 4.2 (2), it is easily checked that $\mathcal{L}_{\tilde{T}}\psi' = 0$. Therefore $L *_1 \psi' = 0$ and the inverse of $*_1\psi'$ under L^{2k-n+2} is given by $C_{n,k} I * *_1\psi' = C_{n,k} *_1 I\psi'$ ([**14**], Prop. 1.2.31).

We have $d\psi' = 0$ (since $\mathcal{L}_T D\omega' = 0$) and thus $d^I\psi = 0$. Applying I to the equation $\tilde{\beta} \wedge \psi' = 0$, we obtain $\tilde{\alpha} \wedge \psi = 0$.

From $0 = i_T(d\alpha \wedge D\omega') = d\alpha \wedge i_T D\omega'$ we deduce that $Lp^* i_T D\omega' = -d\|y\| \wedge p^* D\omega'$ (compare the proof of Lemma 4.2 (1)) and thus $L\psi' = 0$. Since L and $\mathcal{L}_{\tilde{T}}$ commute, $\mathcal{L}_{\tilde{T}}^{2k-n+2} L\psi = L\psi' = 0$. The injectivity of $L^{2k-n+2} : \Omega^{2n-2k-2}(M) \to \Omega^{2k+2}(M)$ and Proposition 4.1 (3) imply that $L\psi = 0$. Since ψ is an n-form, this also implies that $\Lambda\psi = 0$.

Claim 2. There exists a (unique) translation invariant $(k, n-k-1)$-form ω on SV such that $\tilde{\omega} := p^*\omega$ satisfies

$$L\Delta\tilde{\omega} = d\psi. \tag{8}$$

Let us first rewrite this equation. By Claim 1, we have $Ld\psi = dL\psi = 0$. By [**14**], Prop. 1.2.31, Equation (8) is equivalent to

$$\Delta\tilde{\omega} = C_{n,k} * Id\psi. \tag{9}$$

It is easily checked that $*Id\psi$ is a -2-homogeneous form of degree $(k, n-k-1)$. Moreover, $\delta(*Id\psi) = \pm * Id^I d\psi = \pm * Idd^I\psi = 0$.

Let $\tilde{N} := I\tilde{T}$ (in coordinates $\tilde{N} = \sum_{i=1}^n y_i \frac{\partial}{\partial y_i}$).

From Claim 1 we deduce that $\tilde{\alpha} \wedge d\psi = 0$. Taking $*I$ we obtain that $i_{\tilde{N}}(*Id\psi) = 0$ and thus $*Id\psi = \|y\|^{-2} p^* \tau$ for some $(k, n-k-1)$-form τ on SV.

Taking δ and using Lemma 4.2 (3), we obtain

$$0 = \delta(*Id\psi) = \delta(\|y\|^{-2} p^* \tau) = \|y\|^{-2} \delta p^* \tau = \|y\|^{-4} p^* \delta \tau,$$

i.e., $\delta\tau = 0$.

Let us first suppose that $k < n-1$. Then Δ is a bijection on $n - k - 1$-forms on S^{n-1} and therefore there exists a translation invariant $(k, n-k-1)$-form ω with $\Delta\omega = \tau$. Then $\delta d \delta \omega = \delta \Delta \omega = \delta \tau = 0$, which implies $\delta\omega = 0$. We set $\tilde{\omega} := p^* \omega$. From Lemma 4.2 (4) we deduce that $\Delta \tilde{\omega} = \|y\|^{-2} p^* \tau = *Id\psi$. Note also that Lemma 4.2 (3) implies that $\delta\tilde{\omega} = 0$.

If $k = n-1$, then by Theorem 3.3 (3) $D\omega' = f \wedge \alpha \wedge \mu_{S(V)}$ for some function $f \in C^\infty(S(V))$ with

$$\int_{S(V)} y f(y) d\mu_{S(V)}(y) = 0. \tag{10}$$

Equation (10) means that f is orthogonal to the $n-1$-eigenspace of the Laplacian on the $n-1$-dimensional sphere $S(V)$. We can thus solve the equation $\Delta\phi - (n-1)\phi = f$ with $\phi \in C^\infty(S(V))$.

Let $\tilde{f}$ denote the -1-homogeneous extension of f to $V \setminus \{0\}$. We obtain $\psi' = \tilde{f} dy_1 \wedge \ldots \wedge dy_n$ and $\psi = C_{n,k} I \psi' = C_{n,k} \tilde{f} dx_1 \wedge \ldots \wedge dx_n$. Then $*Id\psi = C_{n,k} \sum_{i=1}^n \frac{\partial \tilde{f}}{\partial y_i} *_1 dx_i$.

Let $\tilde{\phi}$ be the 1-homogeneous extension of ϕ to $V \setminus \{0\}$. Then $\Delta\tilde{\phi} = \tilde{f}$ (compare e.g., [**12**] Prop. 4.48). Now set $\tilde{g}_i := \frac{\partial \tilde{\phi}}{\partial y_i}$. These functions are 0-homogeneous and we denote by g_i their restrictions to $S(V)$. Letting $\omega := \sum_{i=1}^n g_i *_1 dx_i$, we compute

$$\Delta\tilde{\omega} = \Delta \sum_{i=1}^n \tilde{g}_i *_1 dx_i = \sum_{i=1}^n \left(\Delta \frac{\partial \tilde{\phi}}{\partial y_i} \right) *_1 dx_i = \sum_{i=1}^n \frac{\partial \tilde{f}}{\partial y_i} *_1 dx_i = C_{n,k} *Id\psi.$$

This finishes the proof of the claim.

Claim 3. $d\omega = D\omega$.

Since d, L and Δ commute,

$$\Delta d(\tilde{\alpha} \wedge d\tilde{\omega}) = -dL\Delta\tilde{\omega} \overset{(8)}{=} -d^2\psi = 0.$$

If $k > 2$, then Lemma 4.3 implies that $d(\tilde{\alpha} \wedge d\tilde{\omega}) = 0$. If $k = 2, n = 3$, we have with the same notations as above $d\tilde{\alpha} \wedge \tilde{\omega} = (dy_1 \tilde{g}_1 + dy_2 \tilde{g}_2 + dy_3 \tilde{g}_3) \wedge *_1 1 = d\tilde{\phi} \wedge *_1 1$ and therefore $d\tilde{\alpha} \wedge d\tilde{\omega} = 0$.

In both cases, it follows that

$$\begin{aligned} 0 &= d\tilde{\alpha} \wedge d\tilde{\omega} \\ &= d(\|y\| p^*\alpha) \wedge d\tilde{\omega} \\ &= d\|y\| \wedge p^*(\alpha \wedge d\omega) + \|y\| p^*(d\alpha \wedge d\omega), \end{aligned}$$

which is only possible if $\alpha \wedge d\omega = 0$, i.e., $d\omega = D\omega$.

Claim 4.

$$*_1^{-1} \Lambda *_1 d\tilde{\omega} = C_{n,k} \psi. \tag{11}$$

From the Kähler identity $[\Lambda, d] = -\delta^I$ one easily obtains $[\Lambda, d^I] = \delta$ and therefore $\delta\psi = [\Lambda, d^I]\psi = 0$.

On the other hand, $L^{2k-n+2} *_1 d\psi = *_1 d\psi' = 0$ and hence $\Lambda *_1 d\psi = 0$.

Now, we compute that

$$\begin{aligned} \delta(*_1^{-1} \Lambda *_1 d\tilde{\omega}) &= *_1^{-1} \Lambda *_1 \Delta\tilde{\omega} \\ &= C_{n,k} *_1^{-1} \Lambda *_1 I * d\psi \qquad \text{by (9)} \\ &= C_{n,k} *_1^{-1} I \Lambda *_1 d\psi \\ &= 0. \end{aligned}$$

From Claim 3 we infer that $Ld\tilde{\omega} = 0$ and, as was shown in Claim 2, $\delta\tilde{\omega} = 0$. We can therefore apply Proposition 4.1 (5) to get $d\psi = L\Delta\tilde{\omega} = C_{n,k} d *_1^{-1} \Lambda *_1 d\tilde{\omega}$.

From these equations we deduce that

$$\Delta *_1^{-1} \Lambda *_1 d\tilde{\omega} = C_{n,k} \Delta\psi.$$

By Lemma 4.3, it follows that

$$*_1^{-1} \Lambda *_1 d\tilde{\omega} = C_{n,k} \psi,$$

provided that $k > 2$.

In the remaining case $k = 2, n = 3$, we use the same notation as in the explicit computation in Claim 2. From $\tilde{\omega} = \sum_{i=1}^3 \tilde{g}_i *_1 dx_i$ we compute that

$$*_1^{-1} \Lambda *_1 d\tilde{\omega} = C *_1^{-1} \Lambda \sum_{i,j=1}^3 \frac{\partial \tilde{g}_i}{\partial y_j} dx_i \wedge dy_j = C *_1^{-1} \sum_{i=1}^3 \frac{\partial \tilde{g}_i}{\partial y_i} = C *_1^{-1} \Delta\tilde{\phi} = C\psi.$$

Claim 5. $\mathfrak{L}^{2k-n} \Psi_\omega = \Psi'$.

From $\mathcal{L}_{\tilde{T}}^{2k-n+3} \psi = \mathcal{L}_{\tilde{T}} \psi' = 0$ and Proposition 4.1 (3), we obtain $\Lambda *_1 \psi = 0$. Now $*_1\psi$ is a $2n - 2k - 2$-form and, by the classical commutator relation on Kähler manifolds, $[\Lambda, L] *_1 \psi = (2k - n + 2) *_1 \psi$.

Therefore $*_1\psi = C_{n,k} [\Lambda, L] *_1 \psi = C_{n,k} \Lambda L *_1 \psi$.

From (11) we get $\Lambda *_1 d\tilde{\omega} = C_{n,k} \Lambda L *_1 \psi$. Therefore $L^{2k-n+1} *_1 d\tilde{\omega} = C_{n,k} L^{2k-n+2} *_1 \psi = C_{n,k} *_1 \psi'$, i.e.,

$$\mathcal{L}_{\tilde{T}}^{2k-n+1} d\tilde{\omega} = C_{n,k} \psi'.$$

Applying Lemma 4.2 (2) we obtain

$$\begin{aligned} C_{n,k}\psi' &= \mathcal{L}_{\tilde{T}}^{2k-n+1} p^* d\omega \\ &= \|y\|^{2k-n+1} p^* \mathcal{L}_T^{2k-n+1} d\omega + d\|y\|^{2k-n+1} \wedge p^* i_T \mathcal{L}_T^{2k-n} d\omega. \end{aligned}$$

Looking at the spherical and the radial part of this equation and using injectivity of p^*, we obtain

$$\mathcal{L}_T^{2k-n+1} d\omega = 0$$
$$i_T D\omega' = C_{n,k} i_T \mathcal{L}_T^{2k-n} d\omega.$$

Since $D\omega' = \alpha \wedge i_T D\omega'$ and $D\mathcal{L}_T^{2k-n}\omega = \alpha \wedge i_T D\mathcal{L}_T^{2k-n}\omega = \alpha \wedge i_T \mathcal{L}_T^{2k-n} d\omega$, we get

$$D\omega' = C_{n,k} D\mathcal{L}_T^{2k-n}\omega.$$

Theorem 2.2 implies that

$$\mathfrak{L}^{2k-n} C_{n,k} \Psi_\omega = \Psi_{C_{n,k}\mathcal{L}_T^{2k-n}\omega} = \Psi_{\omega'} = \Psi'.$$

Therefore, Ψ' is in the image of $\mathfrak{L}^{2k-n}$, which finishes the proof of surjectivity. q.e.d.

Proof of Corollary 0.2. Since G is a subgroup of $O(V)$, it commutes with $\mathfrak{L}$. Part (1) follows from Theorem 2. Alesker proved that $(\mathrm{Val}^{sm}(V))^G$ is finite-dimensional and equals $\mathrm{Val}^G(V)$ if G is compact and acts transitively on $S(V)$ ([**3**], Corollary 1.1.3 and [**6**]). The Lefschetz inequalities thus follow from (1). q.e.d.

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